*Journal of Statistical Physics, Vol. 44, Nos. 5/6, 1986*

# Interacting Walkers on the Cayley Tree, and Polymer Statistics

**V. B. Priezzhev**[1]

*Received July 15, 1985; February 13, 1986*

We obtain the generating function for an ensemble of random walkers on the Cayley tree of coordination number $z$. The pair interaction between walkers is taken into account. This forbids two walkers to occupy the same lattice point after an equal number of steps. Interacting polymer statistics results from this model if one associates time (or the number of steps) with an additional space coordinate. The limiting free energy appears in a form that corresponds to the phase transition of "3/2 order."

**KEY WORDS:** Random walks, Cayley tree, polymer model, phase transition.

## 1. INTRODUCTION

The properties of one-dimensional interacting strings which are embedded in three dimensions are of great importance both in polymer physics and biology. A model that reproduces the configurational properties of hydrocarbon chains inside a lipid membrane has been proposed by Izuyama and Akutsu[1] (to be referred to as IA). This model is a generalization of the two-dimensional model[2] used by Nagle[3] to describe a phase transition in the system of noncontact flexible polymer chains. Polymers or "dislocation lines" in the IA model appear above the critical temperature $T_c$ and may be regarded as directed strings which run vertically through the lattice and do not intersect one another.

IA attempted to prove that the model exhibits a second-order phase transition of the classical type with a jump in the specific heat $C(T)$, i.e., the specific heat is finite as $T \to T_c + 0$ and zero for $T < T_c$. However, Bhattacharjee, Nagle, Huse, and Fisher[4] (see, also Ref. 5) have reconsidered

[1] Joint Institute for Nuclear Research, Laboratory of Theoretical Physics, Moscow, USSR.





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the IA model with a random walk analogy and found that $C(T)$ diverges as $\ln(T-T_c)$ when $T \to T_c+0$ for $d=3$ and is finite for higher dimensions.

The random walk analogy can be elucidated by identification of the vertical $Z$ coordinate with discrete time. An actual question to be solved is a random walk problem of walkers on an $xy$ plane lattice with the restriction that after all walkers have taken the same number of steps, any two of them are not at the same site. When $n=2$ the problem can be solved exactly.[4] The logarithmic law for $C(T)$ follows then from a finite-size scaling ansatz,[4] namely, the supposition that the asymptotic behavior of two walkers remains true for large $n$.

Another approach to this problem has recently been proposed[6] which deals directly with an arbitrary number of walkers. Unfortunately, the sign of contribution to the partition function in this method depends on periodicity of polymers in the vertical direction (assuming periodic boundary conditions). Neglect of the sign difference called "generalized Bethe approximation" leads to the finite jump in $C(T)$. It was noted in Ref. 6 that the method becomes exact if the $xy$ plane lattice has the Bethe structure. The purpose of the present paper is to obtain explicitly the generating function of the above-formulated random walk problem on the Cayley tree.

In Section 2 we use the general method[6] to reduce the original problem to statistics of a single Polya walk. A Polya walk on a Bethe lattice was investigated by Hughes and Sahimi[7] who extended the Montroll generating function formalism[8] to this case and showed that random walks on a Bethe lattice do have some qualitative similarities to random walks on a hypercubic lattice of dimension $d>4$. It is natural to expect that the finiteness of the specific heat at $T_c$ follows from this result according to the analysis of Ref. 4. However, the true answer is quite different. In Section 3 we show that the model exhibits 3/2-order transition in which the specific heat diverges as $(T-T_c)^{-1/2}$. Thus, the IA model on the Bethe lattice demonstrates the two-dimensional behavior[2,3] in spite of apparent multidimensional properties of related random walks.

## 2. GENERAL CONSIDERATIONS

Consider the complete Cayley tree with a coordination number $z$ and a central site 0. Any other site of the lattice is connected with 0 by a unique sequence of bonds. If this sequence consists of $l$ bonds, we assign to the site the coordinate $l$. There are $z(z-1)^{l-1}$ sites with the coordinate $l$ and the total number of sites in the graph is

$$N=z[(z-1)^L-1]/(z-2) \tag{1}$$

where $L$ is the coordinate of boundary sites.

We define an $M$-stepped walk as a connected path along $M$ bonds (perhaps, with repetitions) starting and ending at the same point. The returns to the starting point after 2, 4,..., $M-2$ steps are not prohibited. Two walks do not intersect if they are not at the same point after equal numbers of steps. Denote the statistical weight of one step by $X$. Then the statistical weight of a single $M$-stepped walk $P$ is defined as

$$W(P) = X^M \tag{2}$$

Let $g_n$ be an arbitrary configuration of $n$ nonintersecting $M$-stepped walks on the Cayley tree. The weight of configuration $g_n$ is given by the product

$$\chi(g_n) = \prod_{i=1}^{n} W(P_i) = X^{nM} \tag{3}$$

The problem consists in determining the generation function

$$Z(X) = \sum_{n=0}^{N} \sum_{gn} \chi(g_n) \tag{4}$$

where summation runs over all possible configurations of $n$ $M$-stepped walks with $0 \leqslant n \leqslant N$ and the weight of the void lattice is unity.

The polymer model arises from these definitions if one associates time (or the number of steps after the start) with the spatial $Z$ coordinate. Indeed, the trajectories of walkers moving in a Cayley tree may be regarded as noncontact polymer chains or "dislocation lines" of the IA model. The condition for each $M$-stepped walk to start and to end at the same point means the periodic boundary conditions in $Z$ direction for the obtained 3D lattice. Thus, in the three-dimensional pattern, each $M$-stepped walk is a loop of length $M$ oriented along one spatial direction.

The partition function of the polymer model results from generating function (4) if we attach to the variable $X$ a statistical meaning by setting $X = \exp(-\beta\mu)$, where $\beta$ is the inverse temperature and $\mu$ is a chemical potential of a polymer lino.

Instead of the original problem we consider first a modification of it. Let $P$ be a $K$ walk if it returns to the starting point after $K = kM$ steps for some $k \geqslant 1$ where $k$ is an integer. As above, returns to starting point after 2, 4,..., $K-2$ steps are not prohibited. If $k = 1$, a $K$ walk is an $M$-stepped walk. If $k > 1$, a $K$ walk may contain $K'$-stepped subwalks with $K' = k'M$, $k' < k$. A $K$ walk is said to be nonperiodic if it does not contain two or more coinciding $K$-stepped subwalks.

We introduce the auxiliary functions

$$\bar{W}(P) = (-1)\, X^K \tag{5}$$

for $K$ walks and

$$\bar{\chi}(g_n) = (-1)^n X^{nM} \tag{6}$$

for $M$-stepped walks.

The following proposition holds:

**Theorem.** The product

$$\prod_P [1 + \bar{W}(P)] \tag{7}$$

over all possible nonperiodic $K$ walks $P$ on the Cayley tree equals to the sum over all configurations of $M$-stepped nonintersecting walks including the void lattice

$$\prod_P [1 + \bar{W}(P)] = \sum_{n=0}^{N} \sum_{gn} \bar{\chi}(g_n) \tag{8}$$

The proof of the theorem with more general conditions can be found in Ref. 9. Here we only give the sketch of this proof.

We say that a $K$ walk ($K = kM$, $k > 1$) is self-intersecting if the walker, being at some point at the moment $l$ ($0 \leqslant l < (k-1) M$), visits the same point after $rM$ steps; $r < k$ is an integer. The return to the starting point at the last step of $K$ walks is not self-intersection by the definition. It is the reason why any $M$-stepped walk cannot be self-intersecting.

An essential property of the Cayley tree is that any $K$ walk with $K = kM$, $k > 1$ is self-intersecting.

Let us consider the formal infinite product on the left-hand side of eq. (8), decomposing it into a sum of products of the form $\bar{W}(P_1)\, \bar{W}(P_2) \cdots \bar{W}(P_n)$. If among the set $g_n$ of walks $P_1, ..., P_n$ there are not mutually intersecting and self-intersecting ones, the product equals $\bar{\chi}(g_n)$ and contributes to the right-hand side. And vice versa, each term $\bar{\chi}(g_n)$ in the sum represents the unique product $\bar{W}(P_1) \cdots \bar{W}(P_n)$, where $P_1, ..., P_n$ are different $M$-stepped walks forming the configuration $g_n$.

Consider now a configuration $g_n$ containing two walks $P_i$ and $P_j$ intersecting at some point. Then three exists a configuration $g'_n$ which contains, in place of two walks $P_i$ and $P_j$, a single walk $P'$ with self-intersection at the same point. The first case is described by the term $(-1)\, X^{K_i}(-1)\, X^{K_j}$ in the expansion of (7) and the second by the term $(-1)\, X^{K_i + K_j}$. Therefore, the contributions from mutually intersecting and self-intersecting walk cancel. Similar arguments can be used in the case of several intersection points. Thus only terms of the sum $\sum_g \chi(g)$ survive, where all configurations $g$ consist of solely $M$-stepped walks.

It should be remarked that eq. (8) is very similar to the identity known as Feynman's conjecture and is proved by Sherman[10] to get the combinatorial solution of the planar Ising model.

Equation (8) makes it possible to reformulate the random walk problem of many walkers into a simpler one of a single particle. For this goal it is necessary to establish a relation between the configuration weight $\chi(g)$ and the auxiliary function $\tilde{\chi}(g)$. Let us put $\tilde{X} = Xe^{i\pi/M}$ and note that the change of variables $X \to \tilde{X}$ alters the sign of each $M$-stepped walk in (6). Then $\sum_g \tilde{\chi}(g) \to \sum_g \chi(g)$, and since we assume $M \to \infty$, $Z(X)$ and $Z(\tilde{X})$ coincide because $\tilde{X} \to X$ in the thermodynamic limit. As a result, we may write the generating function in the form

$$Z(X) = \sum_g \tilde{\chi}(g) = \prod_P \left[1 + \bar{W}(P)\right] \tag{9}$$

On the basis of this equation we have

$$\ln Z(X) = \sum_P \ln[1 + \bar{W}(P)] = -\sum_P \sum_{j=1} \frac{[-\bar{W}(P)]^j}{j} \tag{10}$$

We denote by $R_m(i)$ a set of arbitrary $K$ walks which begin anywhere and land on the site $i$ after $m(\text{mod } M)$ steps $0 \leqslant m < M$. The total number of such walks $|R_m(i)|$ obeys, due to the $M$ periodicity in $Z$ direction, the following translation relations

$$|R_0(i)| = |R_1(i)| = \cdots = |R_{M-1}(i)| \tag{11}$$

for each $i$ belonging to the Cayley tree. Writing a sum over all sites and over $M$ times

$$\sum_{m=0}^{M-1} \sum_i |R_m(i)| \tag{12}$$

one can note that each $K$ walk enters into the sum $K$ times if it is non-periodic, and $K/j$ times if it has periodicity $j$. Let $S_K^m(i)$ be the number of $K$ walks with a fixed $K$ in the set $R_m(i)$. Then we can continue eq. (10) regarding $[-\bar{W}(P)]^j$, as a weight of the walk of periodicity $j$

$$-\sum_P \sum_{j=1}^{\infty} \frac{[-\bar{W}(P)]^j}{j} = -\sum_{m=0}^{M-1} \sum_i \sum_K \frac{S_K^m(i)\, X^K}{K}$$
$$= -M \sum_i \sum_K \frac{S_K^0(i)\, X^K}{K} \tag{13}$$

where we convert the summation over nonperiodic $K$ walks in the first sum into the one over arbitrary $K$ walks. The sum over lattice sites can be rearranged due to the symmetry of the Cayley tree. As a result, we obtain for the lattice with a coordination number $z$

$$\ln Z(X) = -M \sum_K \frac{S_K(0)\, X^K}{K} - M \sum_{l=1}^{L} z(z-1)^{l-1} \sum_K \frac{S_K(l)\, X^K}{K} \tag{14}$$

where

$$S_K(l) = S_K^0(i) \tag{15}$$

if the site $i$ has the coordinate $l$.

## 3. SINGLE WALK GENERATING FUNCTION

The considerations in the preceding section lead to the expression (14), which we now make explicit by calculating the sums

$$\sum_K \frac{S_K(l)\, X^K}{K} \qquad l = 0, 1, ..., L \tag{16}$$

including $S_K(l)$—the number of arbitrary close $K$-stepped walks.

Let $W_n(l \mid m)$ be a number of walks starting with coordinate $m$ and terminating after $n$ steps with coordinate $l$. Following the treatment of Hughes and Sahimi[7] we begin with the evolution equation

$$W_{n+1}(l \mid m) = \sum_{l'} \gamma(l, l')\, W_n(l' \mid m) \tag{17}$$

where

$$\begin{aligned} \gamma(l, l') &= (z-1)\, \delta_{l,l'+1} + \delta_{l,l'-1} && 0 < l' < L \\ &= z\delta_{l,l'+1} && l' = 0 \\ &= \delta_{l,l'-1} && l' = L \end{aligned} \tag{18}$$

The origin $l = 0$ and the last shell $l = L$ act as reflecting barriers. Thus the random walks on the Cayley tree can be represented as effective biased walks on the one-dimensional lattice with two "defects." The initial condition is

$$W_0(l \mid m) = \delta_{l,m} \tag{19}$$

To separate translation-invariant and "defect" parts of $\gamma(l, l')$ we write

$$\gamma(l, l') = p(l - l') + q(l, l') \tag{20}$$

where

$$p(l) = (z-1)\,\delta_{l,1} + \delta_{l,-1} \tag{21}$$

and

$$\begin{aligned} q(l, l') &= 0 && l' \neq 0,\ l' \neq L \\ &= \delta_{l,1} - \delta_{l,-1} && l' = 0 \\ &= -(z-1)\,\delta_{l,L+1} && l' = L \end{aligned} \tag{22}$$

Inserting this notation into eq. (17), we obtain

$$W_{n+1}(l \mid m) - \sum_{l} p(l-l')\, W_n(l' \mid m) = \sum_{l'} q(l, l')\, W_n(l' \mid m) \tag{23}$$

It is convenient to introduce a generating function by

$$W(l \mid m; \xi) = \sum_{n=0}^{\infty} W_n(l \mid m)\, \xi^n \tag{24}$$

From eq. (23) using eq. (24) we have

$$W(l \mid m; \xi) - \xi \sum_{l'} p(l-l')\, W(l' \mid m; \xi) = \delta_{lm} + \xi \sum_{l'} q(l, l')\, W(l' \mid m; \xi) \tag{25}$$

A discrete Fourier transform

$$\tilde{W}(\varphi \mid m; \xi) = \sum_{l=-\infty}^{\infty} e^{il\varphi} W(l \mid m; \xi) \tag{26}$$

yields

$$\begin{aligned} \tilde{W}(\varphi \mid m; \xi) = {} & \frac{e^{im\varphi}}{1-\xi\lambda(\varphi)} + \frac{\xi(e^{i\varphi} - e^{-i\varphi})}{1-\xi\lambda(\varphi)}\, W(0 \mid m; \xi) \\ & - \frac{\xi(z-1)\, e^{i(L+1)\varphi}}{1-\xi\lambda(\varphi)}\, W(L \mid m; \xi) \end{aligned} \tag{27}$$

where $\lambda(\varphi)$ is the "structure factor"

$$\lambda(\varphi) = \sum_{l=-\infty}^{\infty} e^{il\varphi} p(l) = (z-1)\, e^{i\varphi} + e^{-i\varphi} \tag{28}$$

Inverting the Fourier transform we find that

$$\begin{aligned} W(l \mid m; \xi) = {} & G(l \mid m; \xi) + \xi W(0 \mid m; \xi)\, H(l; \xi) \\ & - \xi W(L \mid m; \xi)\, F(l; \xi) \end{aligned} \tag{29}$$

with $G(l \mid m; \xi)$, $H(l; \xi)$, $F(l, \xi)$ defined by

$$G(l \mid m; \xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\varphi(m-l)}}{1 - \xi\lambda(\varphi)}\, d\varphi \tag{30}$$

$$H(l; \xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-il\varphi}(e^{i\varphi} - e^{-i\varphi})}{1 - \xi\lambda(\varphi)}\, d\varphi \tag{31}$$

and

$$F(l; \xi) = \frac{(z-1)}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i(L-l+1)\varphi}}{1 - \xi\lambda(\varphi)}\, d\varphi \tag{32}$$

Put $l = 0$ and $l = L$ in eq. (29). We get the system of linear equations

$$\begin{aligned} W(0 \mid m; \xi) &= G(0 \mid m; \xi) + \xi W(0 \mid m; \xi)\, H(0; \xi) \\ &\quad - \xi W(L \mid m; \xi)\, F(0; \xi) \\ W(L \mid m; \xi) &= G(L \mid m; \xi) + \xi W(0 \mid m; \xi)\, H(L; \xi) \\ &\quad - \xi W(L \mid m; \xi)\, F(L; \xi) \end{aligned} \tag{33}$$

which has the solutions

$$W(0 \mid m; \xi) = D^{-1} \det \begin{pmatrix} G(0 \mid m; \xi) & \xi F(0; \xi) \\ G(L \mid m; \xi) & 1 + \xi F(L; \xi) \end{pmatrix} \tag{34}$$

$$W(L \mid m; \xi) = D^{-1} \det \begin{pmatrix} 1 - \xi H(0; \xi) & G(0 \mid m; \xi) \\ -\xi H(L; \xi) & G(L \mid m; \xi) \end{pmatrix} \tag{35}$$

where

$$D \equiv \det \begin{pmatrix} 1 - \xi H(0; \xi) & \xi F(0; \xi) \\ -\xi H(L; \xi) & 1 + \xi F(L; \xi) \end{pmatrix} \tag{36}$$

Up to now we were dealing with the number of arbitrary walks on the Cayley tree. To calculate the sums (16), it is necessary to adapt the general generating function (24) for $M$-stepped walks starting and ending at the same point. For this goal we put in (24)

$$\xi = Xt \exp\left[2\pi i (j/M)\right]$$

The summation over $j$ gives

$$\sum_K S_K(l)\, X^K t^K = \sum_{n=0}^{\infty} \frac{1}{M} \sum_{j=1}^{M} W_n(l \mid l)\, X^n t^n \exp\left(2\pi i \frac{jn}{M}\right) - 1 \tag{37}$$

because only terms with $n=0 \pmod M$ will survive in the right-hand side of eq. (37). Performing the integration over $t$ and changing the summation by integration, i.e., setting

$$\beta=\frac{2\pi j}{M} \qquad d\beta=\frac{2\pi}{M}$$

we obtain for large $M$

$$\sum_K \frac{S_K(l)\,X^K}{K}=\frac{1}{2\pi}\int_0^{2\pi} d\beta \int_0^1 \frac{dt}{t}\,[W(l\mid l; xte^{i\beta})-1] \tag{38}$$

Now, the solution (29) together with eqs. (34) and (35) can be used for deriving thermodynamic properties of the system from the partition function (14).

## 4. THERMODYNAMIC PROPERTIES

In this section we concentrate on the analysis of the partition function near the critical point. The question arises where this point is located. It is shown in Ref. 4 by simple energy–entropy arguments that for the two-dimensional lattice the lowest-lying excited states consist of one $M$-stepped walk. These states have the free energy $kT(-\ln X-\ln z)$, where $z$ is the coordination number of the lattice. They will be thermodynamically preferred to the ground state only when $X>1/z$, which gives the critical point of the model.

Let us begin the analysis by substituting into eq. (38) the first term of the right-hand-side of eq. (29). This yields

$$I_1 \equiv -\frac{1}{2\pi}\int_0^{2\pi} d\beta \int_0^1 \frac{dt}{t}\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-Xte^{i\beta}\lambda(\varphi)}\,d\varphi-1\right] \tag{39}$$

On formally integrating $I_1$ over $t$ we obtain

$$I_1=-\frac{1}{(2\pi)^2}\int_0^{2\pi} d\beta \int_0^{2\pi} d\varphi \ln |1-Xe^{i\beta}[(z-1)\,e^{i\varphi}+e^{-i\varphi}]| \tag{40}$$

which leads at $z=2$ to the known result for the partition function of the two-dimensional polymer model.[2,3] The critical point derived from eq. (40) is $X_c=1/z$. Indeed, the use of Jensen's formula gives

$$I_1=-\frac{1}{2\pi}\int_{|a|>1} \ln |X[(z-1)\,e^{i\varphi}+e^{-i\varphi}]|\,d\varphi \tag{41}$$

where

$$a = X[(z-1)\, e^{i\varphi} + e^{-i\varphi}] \tag{42}$$

Then, $I_1 = 0$ if $X < X_c = 1/z$, and

$$I_1 = -\frac{4}{3\pi}\frac{z^2\sqrt{z}}{[2(z-1)]^{1/2}}(X - X_c)^{3/2} \tag{43}$$

in the vicinity of $X_c$ above $X_c$.

Let us now consider eq. (38) in more detail, taking into account both periodic boundary conditions in the $z$ direction and reflecting boundary conditions in the equivalent one-dimensional lattice. Consider again the first term of eq. (29) defined by eq. (30). Integral (30) is simply evaluated. Denoting

$$d(\xi) = [1 - 4\xi^2(z-1)]^{1/2} \tag{44}$$

and

$$t_{\pm}(\xi- = \frac{1 \pm d(\xi)}{2\xi(z-1)} \tag{45}$$

one can show that

$$\begin{aligned} G(l \mid m; \xi) &= -t_+^{m-l}\theta(1 - |t_+|)/d(\xi) + t_-^{m-l}/d(\xi) \qquad & m \geqslant l \\ &= t_+^{m-l}\theta(|t_+| - 1)/d(\xi) & m < l \end{aligned} \tag{46}$$

Put $l = m$; then $G(l \mid l; \xi) = 0$ for $|\xi| > z^{-1}$. On the other hand, $G(l \mid l; \xi)$ is the generating function of all possible walks returning to the starting point on the unbounded one-dimensional lattice with the structure factor (28). This function can be obtained in a straightforward combinatorial way; the result is

$$G(l \mid l; \xi) = \sum_{K=0}^{\infty} [(z-1)\, \xi^2]^K \frac{(2K)!}{(K!)^2} = \frac{1}{d(\xi)} \tag{47}$$

This result puts into evidence the problem with which we are confronted. The generating function (30) is obtained from the series for nonconstrained walks

$$\tilde{G}(\xi) \equiv 1 + \xi\lambda(\varphi) + [\xi\lambda(\varphi)]^2 + \cdots = \frac{1}{1 - \xi\lambda(\varphi)} \tag{48}$$

by the integration which acts as a filter. But the last identity is true only if $|\xi| \leqslant 1/z$. To deal with the generating functions (30), (31), (32) outside this range, we must analytically continue solution (46), which gives for $|\xi| > 1/z$ expressions of type (47) but not zero. Then, we will obtain

$$\begin{aligned} G(l \mid m; \xi) &= (t_-)^{m-l}/d(\xi) \qquad && m \geqslant l \\ &= (t_+)^{m-l}/d(\xi) && m < l \end{aligned} \tag{49}$$

$$\begin{aligned} H(l; \xi) &= (t_- - t_{\mp}^{-1})/d(\xi) \qquad && l = 0 \\ &= t_{\mp}^{-l}(t_+ - t_{\mp}^{-1})/d(\xi) && l \geqslant 1 \end{aligned} \tag{50}$$

$$F(l; \xi) = (z-1)\, t^{L-l+1}/d(\xi) \tag{51}$$

Formula (38) with functions (49), (50), (51) no longer gives the critical point $x_c = 1/z$. The location of $X_c$ is determined now by zeros of $d(\xi)$ at $\xi_c = \pm 2^{-1}(z-1)^{-1/2}$. The function $W(l \mid l; \xi)$ diverges at $\xi_c$. Its critical behavior can be found if one treats $W(l \mid l; \xi)$ as a generating function of one-dimensional unbiased random walks that start at the point $l$ of the interval $L$ and return to the same point. Indeed, for each step to the right with the weight $\xi(z-1)$, there is a step to the left weighted by $\xi$. So, a pair of steps has the weight $\xi^2(z-1)$, which gives the single step weight $\xi' = \xi(z-1)^{1/2}$ for the effective unbiased walk. The calculation of this generating function is a standard random walk problem.[11] For slightly changed boundary conditions (not sufficient in the thermodynamical limit) the answer is

$$W(l \mid l; \xi') = \frac{1}{L} \sum_{r=1}^{L} \frac{1 + \cos[\pi r(2j-1)/L]}{1 - 2\xi' \cos(\pi r/L)} \tag{52}$$

or, for large $L$

$$W(l \mid l; \xi') = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{1}{1 - 2\xi' \cos\varphi} + \frac{\cos[\varphi(2j-1)]}{1 - 2\xi' \cos\varphi} \right\} d\varphi \tag{53}$$

Now, we must put $\xi' = \xi(z-1)^{1/2}$. The second integral in (53) decreases exponentially with $j$ and does not contribute to the free energy. The substitution of the first one into eq. (38) gives

$$I = -\frac{1}{2\pi} \int_0^{2\pi} d\beta \int_0^1 \frac{dt}{t} \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{1 - 2Xt(z-1)^{1/2} e^{i\beta} \cos\varphi} - 1 \right] d\varphi \tag{54}$$

Thus we came again to eq. (39) with $z = 2$ and the new critical point $X_c = 2^{-1}(z-1)^{-1/2}$. Proceeding as above we get $I = 0$ if $X < X_c$ and

$$I = -(16/3\pi)(z-1)^{3/4}\,(X - X_c)^{3/2} \tag{55}$$

above $x_c$. The free energy of the model is given by eqs. (14), (38), and (55). In the case $z=2$, eqs. (43) and (55) coincide, but for $z>2$ we obtain another $z$ dependence of thermodynamic quantities and another critical point $X_c$.

The obtained 3/2-order transition calls for comment. It was noted in the Introduction that the phase transition of this type occurs in two-dimensional polymer models.[2,3] In our notation it is the case $z=2$. There is a drastic difference between random walk behavior for $z=2$ and $z>2$. For any pair of walkers in the former case, its coordinates $l_1$ and $l_2$ are strongly ordered, say $l_1>l_2$, at any moment of time, whereas in the latter, permutations of $l_1$ and $l_2$ are permitted. The sole restriction on a walk configuration for $z>2$ is the absence of $K$-stepped walks with the period $k=K/M>1$. Nevertheless, our results show that this reduced constraint is still too strong to give the logarithmic singularity of the partition function.

## ACKNOWLEDGMENT

I would like to thank Dr. C. S. O. Yokoi, whose remarks stimulated a more detailed analysis of the obtained solutions.

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